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THEORY AND NUMERICAL ANALYSIS OF SHELLS UNDERGOING LARGE ELASTIC STRAINS

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Abstract—A non-linear bending theory of rubber-like shells undergoing large elastic strains is proposed. The theory is based on a relaxed normality hypothesis and the incompressibility condition. Using series expansion in the normal direction and applying some estimation technique a consistent first approximation and a simplest approximation to the elastic strain energy of the shell are constructed. Lagrangian displacement shell equations are derived and the incremental shell deformation is considered. The numerical results presented for one- and two-dimensional large strain shell problems confirm the accuracy and efficiency of the proposed shell theory.

1. INTRODUCTION

The classical geometrically non-linear theory of thin isotropic elastic shells as developed by Aron, Love, Chien, Mushtari, Galimov, Alumyae and Koiter, to mention only a few veteran names, has reached a certain degree of completeness in the last two decades. We refer to a comprehensive review by Pietraszkiewicz (1989), where extensive references are given. Within this classical theory the adequate description of shell deformation can be based on the simple kinematic hypothesis that normals to the undeformed reference surface remain normals and inextensible during deformation. This allows the transverse shearing and normal strains to be ignored from the outset in the description of shell kinematics. However, the effect of change in the shell thickness due to the normal strains is taken into account in the constitutive equations by using an independent static hypothesis about the approximately plane stress state in the shell. As a result, the two-dimensional surface strain energy, being a sum of two quadratic functions describing the stretching and bending of the shell reference surface, is the first consistent approximation to the three-dimensional shell strain energy [see Koiter (1960)]. Based on this classical theory efficient computer programs were developed and applied to analyze a number of highly non-linear problems of one- and two-dimensional deformation of thin elastic shells within the full unrestricted range of finite displacements and rotations [see e.g. Nolte (1983) and Nolte *et al.* (1986)]. During the past few years shell theories with additional rotational degrees of freedom had been presented, accounting also for shear deformation [see e.g. Gruttmann *et al.* (1989), Başar and Ding (1990) and Simo *et al.* (1990)].

The aim of this paper is to derive a non-linear bending theory of rubber-like shells undergoing large elastic strains. Contrary to the classical small-strain theory, in this case the effect of transverse normal strains becomes of primary importance and should be taken into account not only in the two-dimensional constitutive equations, but also in the description of the shell kinematics. In the first papers devoted to this problem it was usually assumed that normals remain normals but are allowed to change their length. The uniform deformation in the normal direction due to stretching of the shell middle surface was taken into account by Biricikoglu and Kalnins (1971), while Chernykh (1980, 1983) also considered the quadratic normal deformation due to the stretching and bending of the middle surface. These results were extended by Stumpf and Makowski (1986) who proved, in particular, that within such a modified normality hypothesis the incompressibility condition allows the determination of the non-linear distribution of strains in the normal direction, in closed form. On the other hand, Simmonds (1985) assumed from the outset that the state of a thin shell is entirely described by the change of the mid-surface metric and curvature tensors. With the help of invariance requirements it was then shown that the corresponding

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surface strain energy function should be quadratic in the bending strains and of the same form as for a flat plate.

A more general theory for shells undergoing large rotations and large strains was proposed by Makowski and Stumpf (1986, 1989, 1990). Using on one side equilibrium equations and statical boundary conditions derived by descent from the three-dimensional continuum and on the other side the corresponding kinematical model of a Cosserat surface with three deformable directors, the theory is statically and kinematically exact. Analogously to Stumpf and Makowski (1986) the incompressibility condition is used to determine the non-linear distribution of strains through the shell thickness in closed form. Approximations are entered into the theory by reducing the three-dimensional constitutive equations to a two-dimensional form. Applications are considered for axisymmetric deformations of shells of revolution undergoing large strains (Makowski and Stumpf, 1989). Finally it should be mentioned that the shell model of Simo *et al.* (1990) accounts for a uniform deformation in the transverse direction.

It is not apparent, however, that during an arbitrary deformation of rubber-like shells the initially normal material fibres remain normal or straight, as it is assumed in the papers cited above. All that follows from the known one-dimensional solutions given, for example, by Taber (1987a, b), Libai and Simmonds (1988) and Makowski and Stumpf (1989) is that the transverse shearing strains are usually small and only near clamped and loaded edges or near point loads they have to be taken into account [see e.g. Taber (1987b)]. Their influence on the solution can be ignored in a wide range of parameters. Therefore, in this paper we use a relaxed normality hypothesis (6), according to which material fibres, initially straight and normal to any undeformed surface parallel to the reference surface, remain normal to it during the shell deformation (see Fig. 1). This hypothesis allows us to ignore, from the outset, the transverse shearing strains, but does not put any constraints on the extension and bending of the initially normal material fibres.

The behaviour of most rubber-like materials undergoing finite strains is such that they allow only for an isochoric (volume preserving) or almost isochoric deformation, e.g. Green and Adkins (1960) and Green and Zerna (1968). Therefore, in this paper we freely use the incompressibility condition (7).

The short discussion given in Chapter 3 shows that the range of shell problems in which bending effects may become important is limited to shells of at most moderate initial or/and deformed thickness, undergoing at most moderate bending strains, while the membrane strains should not exceed the order of unity (see eqns (13) and (14)).

Within the constraints and the range of parameters described above, the three-dimensional Green strain tensor and the corresponding shell strain energy density are expanded into series in the normal direction, and orders of all terms in the series are estimated in Chapters 4 and 5. As a result, by ignoring all small terms the two-dimensional consistent first approximation (40) and the simplest approximation (47) to the elastic strain energy of rubber-like shells are constructed and used to derive in Chapter 6 corresponding constitutive equations. It is interesting to note that both approximations are expressed only in terms of stretching and bending of the reference surface. The structure of our derived constitutive equations in the case of thin shells seems to be simpler than the one suggested by Simmonds (1985). For the special case of cylindrical deformation of shells our first approximation (40)

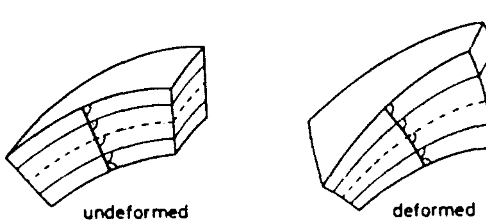

Fig. 1. Undeformed and deformed shell element.

leads to the strain energy density of Libai and Simmonds (1981) and for shells of revolution with axisymmetric deformations the simplest approximation (47) corresponds to the result of Simmonds (1986).

In Chapter 7 the Lagrangian displacement shell equations are derived and in Chapter 8 the incremental formulation of the large strain shell theory is considered. In Chapter 9 we present the numerical results for several one- and two-dimensional large strain shell problems. A comparison of our one-dimensional results with those reported in the literature shows the accuracy and efficiency of the presented shell theory. Results for two-dimensional problems are not yet available in the literature.

2. NOTATION AND GEOMETRIC RELATIONS

In this paper we shall follow, where it is possible, the notation scheme used by Koiter (1966), Naghdi (1972), Pietraszkiewicz (1977, 1979, 1983, 1989), Stumpf (1981, 1986) and Schieck (1989).

Let $\mathfrak{P}$ be the region of three-dimensional Euclidean point space occupied by the shell in the undeformed configuration. In $\mathfrak{P}$ the normal system of curvilinear coordinates $\{\theta^\alpha, \zeta\}$, $\alpha = 1, 2$, is introduced such that $-h/2 \leqslant \zeta \leqslant +h/2$ is the distance from the middle surface $\mathfrak{M}$ of $\mathfrak{P}$ and h is the undeformed shell thickness.

The geometry of the undeformed shell middle surface $\mathfrak{M}$ is described through its position vector $\mathbf{r} = \mathbf{r}(\theta^\alpha)$. At each point $M \in \mathfrak{M}$ we define the natural surface base vectors $\mathbf{a}_\alpha = \partial \mathbf{r}/\partial \theta^\alpha \equiv \mathbf{r}_{,\alpha}$, the unit normal vector $\mathbf{n} = \frac{1}{2}\varepsilon^{\alpha\beta}\mathbf{a}_\alpha \times \mathbf{a}_\beta$ as well as the covariant components of the surface metric tensor $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$, of the surface curvature tensor $b_{\alpha\beta} = -\mathbf{a}_\alpha \cdot \mathbf{n}_{,\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{n}$ and of the surface permutation tensor $\varepsilon_{\alpha\beta} = (\mathbf{a}_\alpha \times \mathbf{a}_\beta) \cdot \mathbf{n}$ such that $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = \sqrt{a}$ where $a = |a_{\alpha\beta}|$. The reciprocal surface base vectors $\mathbf{a}^\beta$ and contravariant components of the surface metric tensor $a^{\alpha\beta}$ are then defined by $\mathbf{a}^\beta \cdot \mathbf{a}_\alpha = \delta^\beta_\alpha$ and $a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta$, respectively, where δ^α_β is the Kronecker symbol. A dot indicates the scalar product and the symbol $\times$ the cross product of two vectors.

The geometry of the undeformed shell space $\mathfrak{P}$ is now described in the convective coordinates $\{\theta^i\}$, $i = 1, 2, 3$, $\theta^3 \equiv \zeta$, through the position vector $\mathbf{p}(\theta^i) = \mathbf{r}(\theta^\alpha) + \zeta\mathbf{n}(\theta^\alpha)$. At each point $P \in \mathfrak{P}$ we have the natural base vectors $\mathbf{g}_i = \mathbf{p}_{,i}$, the covariant components of the metric tensor $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ and of the permutation tensor $\varepsilon_{ijk} = (\mathbf{g}_i \times \mathbf{g}_j) \cdot \mathbf{g}_k$ with $g = |g_{ij}|$, the reciprocal base vectors $\mathbf{g}^j$ satisfying $\mathbf{g}^j \cdot \mathbf{g}_i = \delta^j_i$ and the contravariant components $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$ of the metric tensor. These spatial quantities are related to the surface quantities by

$$\mathbf{g}_\alpha = \mu^\lambda_\alpha \mathbf{a}_\lambda, \quad \mathbf{g}_3 = \mathbf{n}, \quad g_{\alpha\beta} = \mu^\lambda_\alpha \mu^\mu_\beta a_{\lambda\mu}, \quad g_{\alpha 3} = 0, \quad g_{33} = 1,$$
$$\mathbf{g}^\alpha = (\mu^{-1})^\alpha_\lambda \mathbf{a}^\lambda, \quad \mathbf{g}^3 = \mathbf{n}, \quad g^{\alpha\beta} = (\mu^{-1})^\alpha_\lambda (\mu^{-1})^\beta_\mu a^{\lambda\mu}, \quad g^{\alpha 3} = 0, \quad g^{33} = 1, \tag{1}$$

where the shifter μ^λ_α and its inverse $(\mu^{-1})^\alpha_\lambda$ are given by

$$\mu^\lambda_\alpha = \delta^\lambda_\alpha - \zeta b^\lambda_\alpha, \quad (\mu^{-1})^\alpha_\lambda \mu^\lambda_\beta = \delta^\alpha_\beta, \quad (\mu^{-1})^\alpha_\lambda = \frac{1}{\mu}[\delta^\alpha_\lambda - \zeta(2H\delta^\alpha_\lambda - b^\alpha_\lambda)],$$

$$\mu = |\mu^\beta_\alpha| = \sqrt{\frac{g}{a}} = 1 - \zeta 2H + \zeta^2 K, \quad H = \tfrac{1}{2}b^\alpha_\alpha, \quad K = \tfrac{1}{2}\varepsilon^{\alpha\lambda}\varepsilon_{\beta\mu} b^\beta_\alpha b^\mu_\lambda. \tag{2}$$

Consider now an arbitrary smooth deformation $\chi : \mathfrak{P} \to \bar{\mathfrak{P}}$, $\bar{\mathbf{p}}(\theta^i) = \chi[\mathbf{p}(\theta^i)]$, of the shell described in the convective coordinate system $\{\theta^i\}$. The geometry of the deformed shell space $\bar{\mathfrak{P}}$ is now described by geometric quantities analogous to those describing $\mathfrak{P}$, only now marked by an overbar: $\bar{\mathbf{g}}_i$, $\bar{g}_{ij}$, $\bar{\mathbf{g}}^i$, $\bar{g}^{ij}$, etc. Introducing the deformation gradient tensor

$$\mathbf{F} = \nabla\chi = \bar{\mathbf{g}}_i \otimes \mathbf{g}^i, \quad \bar{\mathbf{g}}_i = \mathbf{F}\mathbf{g}_i, \tag{3}$$

where $\otimes$ indicates the tensor product of two vectors, the strains in the shell space are described either by the right stretch tensor $\mathbf{U} = (\mathbf{F}^T\mathbf{F})^{1/2}$ or by the Green strain tensor

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^T\mathbf{F}-\mathbf{1}) = E_{ij}\mathbf{g}^i \otimes \mathbf{g}^j, \quad E_{ij} = \tfrac{1}{2}(\bar{g}_{ij}-g_{ij}), \tag{4}$$

where $\mathbf{1}$ is the metric tensor of $\mathfrak{P}$.

During the deformation χ the shell middle surface moves from $\mathfrak{M}$ to $\bar{\mathfrak{M}}$ defined by $\bar{\mathbf{r}}(\theta^\alpha) = \chi[\mathbf{r}(\theta^\alpha)]$. The geometry of $\bar{\mathfrak{M}}$ is described by quantities analogous to those describing $\mathfrak{M}$, only now marked by an overbar: $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$, $\bar{a}_{\alpha\beta}$, $\bar{b}_{\alpha\beta}$, $\bar{a}$, $\bar{\mathbf{a}}^\beta$, $\bar{a}^{\alpha\beta}$, $\bar{H}$, $\bar{K}$, etc. The Lagrangian surface strain tensor $\gamma = \gamma_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta$ and the tensor of change of surface curvature $\boldsymbol{\kappa} = \kappa_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta$ are usually defined by

$$\gamma_{\alpha\beta} = \tfrac{1}{2}(\bar{a}_{\alpha\beta}-a_{\alpha\beta}), \quad \kappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta}-b_{\alpha\beta}). \tag{5}$$

They are functions of the displacement field $\mathbf{u}$ through $\bar{\mathbf{r}} = \mathbf{r}+\mathbf{u}$.

3. BASIC ASSUMPTIONS

In order to simplify subsequent transformations we ignore the transverse shearing strains by assuming from the outset that $E_{\alpha 3} = 0$, or equivalently

$$\bar{\mathbf{g}}_\alpha \cdot \bar{\mathbf{g}}_3 \equiv \bar{g}_{\alpha 3} = 0. \tag{6}$$

We call this assumption the relaxed normality hypothesis. Indeed (see Fig. 1), the hypothesis requires the material fibres, initially straight and normal to any undeformed surface Ξ defined by ζ = const., to remain always normal to it during the shell deformation. Please note that (6) does not put any constraints on the extension and bending of the initially normal material fibres.

In what follows we also assume that during the shell deformation the following incompressibility condition

$$\det \mathbf{F} = \sqrt{\frac{\bar{g}}{g}} = 1 \tag{7}$$

is satisfied, where

$$\frac{\bar{g}}{g} = \tfrac{1}{6}\varepsilon^{ijk}\varepsilon_{lmn}(\delta_i^l+2E_i^l)(\delta_j^m+2E_j^m)(\delta_k^n+2E_k^n). \tag{8}$$

Under the relaxed normality hypothesis (6) we can reduce (8) to

$$\frac{\bar{g}}{g} = \lambda_3^2\frac{\tilde{g}}{g}, \quad \lambda_3^2 = \bar{\mathbf{g}}_3\cdot\bar{\mathbf{g}}_3 = 1+2E_{33}, \tag{9}$$

$$\frac{\tilde{g}}{g} = 1+2E_\alpha^\alpha+2(E_\alpha^\alpha E_\beta^\beta-E_\beta^\alpha E_\alpha^\beta), \quad \tilde{g} = |\bar{g}_{\alpha\beta}|. \tag{10}$$

If we introduce the unit vector $\bar{\mathbf{m}}$ such that

$$\bar{\mathbf{g}}_3 = \lambda_3(\zeta)\bar{\mathbf{m}}(\zeta), \tag{11}$$

it follows from (9) that $\lambda_3 = \lambda_3(\zeta)$ describes the stretching of those material fibres which are initially normal to $\mathfrak{M}$. Under the additional incompressibility condition (7) the stretching λ_3 is uniquely defined through $E_{\alpha\beta}$ by

$$\lambda_3 = \frac{1}{\sqrt{1+2E^\alpha_\alpha+2(E^\alpha_\alpha E^\beta_\beta - E^\beta_\alpha E^\alpha_\beta)}}. \tag{12}$$

Under the assumptions (6) and (7), the non-linear theory of shells to be discussed in this paper should properly describe the shell deformation caused by the stretching and bending of its reference surface and by transverse normal strains according to (12). Let γ and κ be the greatest eigenvalues of the surface strain measures $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$, while R and $\bar{R}$ be the smallest principal radii of curvature of $\mathfrak{M}$ and $\bar{\mathfrak{M}}$, respectively. Let θ be a small parameter such that $\theta^2 \ll 1$, i.e. $1+\theta^2 \simeq 1$. It is commonly accepted that a shell theory can be applied only to such problems of shell-like solids in which neither h and R nor $\bar{h}$ and $\bar{R}$ are of comparable order, where h and $\bar{h}$ are the shell thickness in the undeformed and deformed configuration, respectively. It is also known that if the deformation is finite, i.e. $\gamma > 0(1)$, where 0( ) stands for "of the order of", then for sufficiently thin shells bending effects can be ignored and the simple membrane theory can be applied, as in Green and Adkins (1960). Therefore, the range of shell problems in which the bending effects may become important is limited by $\gamma \leqslant 0(1)$. Within this range it follows from (12) that $\bar{h} = h[1+0(\gamma)] = 0(h)$. Moreover, since $\kappa = 0(1/\bar{R}+1/R)$, the bending strains in the shell space can reach the order $h\kappa = 0(\theta)$ provided one of h/R, $h/\bar{R}$ or both are $0(\theta)$. Therefore, in the subsequent derivation of non-linear relations of shells, we assume the following ranges of admissible values of the parameters:

$$\frac{h}{R} \leqslant 0(\theta), \quad \frac{h}{\bar{R}} \leqslant 0(\theta), \tag{13}$$

$$\gamma \leqslant 0(1), \quad h\kappa \leqslant 0(\theta). \tag{14}$$

4. SHELL STRAIN MEASURES

Since by (12) and (9) E_{33} is entirely determined by $E_{\alpha\beta}$, let us introduce the tensor $\tilde{\mathbf{E}} = E_{\alpha\beta}\mathbf{g}^\alpha \otimes \mathbf{g}^\beta$ defined on the surface $\mathfrak{S}$ and expand it into series with respect to the normal coordinate

$$\tilde{\mathbf{E}}(\zeta) = \sum_{n=0}^{\infty} \frac{1}{n!}\zeta^n \mathbf{E}_{(n)} \tag{15}$$

with

$$\mathbf{E}_{(0)} = \tilde{\mathbf{E}}|_{\zeta=0} \equiv \boldsymbol{\gamma} = \gamma_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta, \tag{16}$$

$$\mathbf{E}_{(n)} = \frac{\partial^n \tilde{\mathbf{E}}}{(\partial\zeta)^n}\bigg|_{\zeta=0} = E_{\alpha\beta;33\ldots3}|_{\zeta=0}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta, \quad n \geqslant 1; \tag{17}$$

$$\mathbf{E}_{(1)} \equiv \boldsymbol{\chi} = \chi_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta, \quad \mathbf{E}_{(2)} \equiv 2\boldsymbol{\pi} = 2\pi_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta, \tag{18}$$

where $(\ )_{;3}$ denotes the covariant derivative in the ζ-direction with respect to the metric $g_{\alpha\beta}$.

Introducing the curvature tensor of $\mathfrak{S}$ by

$$h_{\alpha\beta} = -\mathbf{g}_{3,\alpha}\cdot\mathbf{g}_\beta = \mathbf{g}_3\cdot\mathbf{g}_{\alpha,\beta} = b^\lambda_\alpha \mu_{\lambda\beta} = -\tfrac{1}{2}g_{\alpha\beta,3}, \tag{19}$$

$$h_{\alpha\beta,3} = -b^\lambda_\alpha b_{\lambda\beta}, \quad h^\beta_{\alpha,3} = h^\lambda_\alpha h^\beta_\lambda, \tag{20}$$

for two first covariant derivatives in (17) we obtain

$$E_{\alpha\beta;3} = E_{\alpha\beta,3} + h_\alpha^\lambda E_{\lambda\beta} + h_\beta^\lambda E_{\lambda\alpha}, \tag{21}$$

$$E_{\alpha\beta,33} = E_{\alpha\beta,33} + 2h_\alpha^\lambda E_{\lambda\beta,3} + 2h_\beta^\lambda E_{\alpha\lambda,3} - 2h_\alpha^\lambda h_\beta^\rho E_{\lambda\rho}. \tag{22}$$

Defining the curvature tensor $\bar{h}_{\alpha\beta}$ of $\bar{\mathfrak{S}}$ analogously to (19) we obtain under the relaxed normality hypothesis

$$\bar{\mathbf{g}}_{\alpha,3}\cdot\bar{\mathbf{g}}_\beta = -\lambda_3\bar{h}_{\alpha\beta}, \quad \bar{\mathbf{g}}_{3,3}\cdot\bar{\mathbf{g}}_\alpha = -\lambda_3\lambda_{3,\alpha}, \quad \bar{\mathbf{g}}_{3,3}\cdot\bar{\mathbf{g}}_3 = \lambda_3\lambda_{3,3}, \tag{23}$$

which allows us to find

$$\begin{aligned}\bar{h}_{\alpha\beta,3} &= -(\bar{\mathbf{m}}_{,\alpha}\cdot\bar{\mathbf{g}}_\beta)_{,3} \\ &= -(\bar{\mathbf{m}}_{,3}\cdot\bar{\mathbf{g}}_\beta)_{,\alpha} + \bar{\mathbf{m}}_{,3}\cdot\bar{\Gamma}^\iota_{\alpha\beta}\bar{\mathbf{g}}_\iota + \bar{h}_\alpha^\lambda\bar{\mathbf{g}}_\lambda\cdot\bar{\mathbf{g}}_{3,\beta} \\ &= \lambda_{3,\alpha|\beta} - \lambda_3\bar{h}_\alpha^\lambda\bar{h}_{\lambda\beta}\end{aligned} \tag{24}$$

where $\bar{\mathbf{m}} = (\bar{\mathbf{g}}_1\times\bar{\mathbf{g}}_2)/\sqrt{\bar{g}}$ is the unit normal vector on $\bar{\mathfrak{S}}$ (see also eqn (11)), and $(\ )_{|\beta}$ denotes the surface covariant differentiation in the metric $\bar{g}_{\alpha\beta}$.

Now with the help of (19)–(24) we obtain for $\chi_{\alpha\beta}$ and $\pi_{\alpha\beta}$

$$\chi_{\alpha\beta} = -(\lambda\bar{h}_{\alpha\beta} - h_{\alpha\beta}) + h_\alpha^\lambda\gamma_{\lambda\beta} + h_\beta^\lambda\gamma_{\lambda\alpha}, \tag{25}$$

$$2\pi_{\alpha\beta} = \lambda^2\bar{h}_\alpha^\lambda\bar{h}_{\lambda\beta} - h_\alpha^\lambda h_{\lambda\beta} - \lambda\lambda_{,\alpha\|\beta} - \rho\bar{h}_{\alpha\beta} + 2h_\alpha^\lambda\chi_{\lambda\beta} + 2h_\beta^\lambda\chi_{\alpha\lambda} - 2h_\alpha^\lambda h_\beta^\rho\gamma_{\lambda\rho}, \tag{26}$$

where with the help of (12) two surface invariants have been introduced

$$\lambda = \lambda_{3|\zeta=0} = \sqrt{\frac{a}{\bar{a}}} = \frac{1}{\sqrt{1+2\gamma_\alpha^\alpha + 2(\gamma_\alpha^\alpha\gamma_\beta^\beta - \gamma_\beta^\alpha\gamma_\alpha^\beta)}}, \tag{27}$$

$$\rho = \lambda_{3,3|\zeta=0} = -\lambda^3[\chi_\alpha^\alpha + 2(\gamma_\alpha^\alpha\chi_\beta^\beta - \gamma_\beta^\alpha\chi_\alpha^\beta)]. \tag{28}$$

Note that according to (27) $\lambda = \lambda(\gamma_{\alpha\beta})$, and from (25), (27) and (5) it follows that $\chi_{\alpha\beta} = \chi_{\alpha\beta}(\gamma_{\lambda\mu}, \kappa_{\lambda\mu})$. Therefore, it is apparent from (25)–(28) and (5) that $\rho = \rho(\gamma_{\alpha\beta}, \kappa_{\alpha\beta})$ and $\pi_{\alpha\beta} = \pi_{\alpha\beta}(\gamma_{\lambda\mu}, \kappa_{\lambda\mu})$.

For higher order terms of the expansion (15) it is possible to prove [see Schieck (1989)] the following estimates

$$\frac{\|E_{(n+1)}\|}{\|E_{(n)}\|} = 0\left(\frac{1}{R}, \frac{1}{\bar{R}}, \frac{1}{L}\right), \quad n \geq 1, \tag{29}$$

where L is the smallest wavelength of deformation pattern on $\mathfrak{M}$ allowing the estimation of orders of the surface covariant derivatives in the metric $a_{\alpha\beta}$ by $\|(\ )_{|\alpha}\|/\|(\ )\| = 0(1/L)$.

Let us define the small parameter θ by

$$\theta = \max\left(\frac{h}{R}, \frac{h}{\bar{R}}, \frac{h}{L}\right). \tag{30}$$

Using estimate (29) and eqns (12) and (13) it follows from (15) that within the relative error $0(\theta^2)$ in the bending strains χ the strain tensor $\tilde{\mathbf{E}}$ of the surface $\tilde{\mathfrak{S}}$ can be approximated by the following quadratic expression

$$\tilde{\mathbf{E}} = \gamma + (\zeta\chi + \zeta^2\pi)[1 + 0(\theta^2)]. \tag{31}$$

This approximation is compatible with the assumptions (6) and (7).

5. ELASTIC STRAIN ENERGY OF THE SHELL

The constitutive equations of an incompressible isotropic solid are described by the three-dimensional strain energy density $W(\mathbf{E}) = W(I_1, I_2)$, defined per unit volume of $\mathfrak{P}$, as the function of the two standard strain invariants I_1 and I_2 with $I_3 = 1$ [e.g. Green and Zerna (1968)]. Since by (12) and (9) E_{33} is entirely determined by $E_{\alpha\beta}$, we can always express E_{33} as a function of $E_{\alpha\beta}$ and obtain $\tilde{W}(\tilde{\mathbf{E}})$ from W. This is a consequence of the assumed incompressibility of the material, but it should be pointed out that it does not exclude hydrostatic pressure or transverse normal stress.

The shell strain energy function, per unit area of $\mathfrak{M}$, can be defined by

$$\Phi = \int_{-h/2}^{+h/2} \tilde{W}\mu\delta\zeta. \tag{32}$$

Let us expand the integrand of (32) into series with respect to ζ and integrate it through the thickness to obtain

$$\begin{aligned}\Phi = h\Big\{ & W_{(0)}\left(1+\frac{h^2}{12}K\right)+\mathbf{W}_{(1)}\cdot\left[-\frac{h^2}{12}b^\lambda_\lambda\chi+\left(\frac{h^2}{24}+\frac{h^4}{160}K\right)2\pi\right.\\ & \left.-\frac{h^4}{480}b^\lambda_\lambda\mathbf{E}_{(3)}+\left(\frac{h^4}{1920}+\frac{h^6}{10752}K\right)\mathbf{E}_{(4)}-\cdots\right]\\ & +\mathbf{W}_{(2)}\cdot\left[\left(\frac{h^2}{24}+\frac{h^4}{160}K\right)\chi\otimes\chi-\frac{h^4}{160}b^\lambda_\lambda\chi\otimes 2\pi\right.\\ & \left.+\left(\frac{h^4}{1920}+\frac{h^6}{10752}K\right)(12\pi\otimes\pi+4\chi\otimes\mathbf{E}_{(3)})-\cdots\right]\\ & +\mathbf{W}_{(3)}\cdot\left[-\frac{h^4}{480}b^\lambda_\lambda\chi\otimes\chi\otimes\chi+\left(\frac{h^4}{320}+\frac{h^6}{1792}K\right)\chi\otimes\chi\otimes 2\pi-\cdots\right]\\ & +\mathbf{W}_{(4)}\cdot\left[\left(\frac{h^4}{1920}+\frac{h^6}{10752}K\right)\chi\otimes\chi\otimes\chi\otimes\chi-\cdots\right]+\cdots\Big\},\end{aligned} \tag{33}$$

where

$$W_{(0)} = \tilde{W}[\tilde{\mathbf{E}}(\zeta)]|_{\zeta=0}, \quad \mathbf{W}_{(n)} = \left.\frac{\partial^n \tilde{W}[\tilde{\mathbf{E}}(\zeta)]}{(\partial\tilde{\mathbf{E}})^n}\right|_{\zeta=0}, \quad n \geqslant 1. \tag{34}$$

The range of values that can be taken by derivatives (34) is determined by the material law and by the range of admissible membrane strains (14). However, all we need in order to simplify (33) is to estimate orders of magnitudes that might be taken into account by (34) and not their precise values. This can be done on the basis of the following approximate considerations.

Let us take, as an example, the incompressible Mooney material [see Mooney (1940) and Green and Adkins (1960)] for which

$$W = C_1(I_1-3)+C_2(I_2-3), \tag{35}$$

$$I_1 = \lambda_1^2+\lambda_2^2+\frac{1}{\lambda_1^2\lambda_2^2}, \quad I_2 = \frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\lambda_1^2\lambda_2^2, \tag{36}$$

$$\lambda_\alpha^2 = 1+2E_\alpha, \quad 2(C_1+C_2) = G, \tag{37}$$

where λ_α are principal stretches, E_α are principal Green strains and G is the shear modulus.

Let λ_1 be the larger principal stretch value. Then within the admissible $\gamma \leqslant 0(1)$ estimates for $\mathbf{W}_{(n)}$ can be obtained by reducing the problem to the one-dimensional case, i.e. assuming $\lambda_2 \equiv 1$, which gives

$$W = 2G\frac{E_1^2}{1+2E_1}, \quad \frac{dW}{dE_1} = 4G\frac{E_1+E_1^2}{1+2E_1^2}, \quad \frac{d^nW}{(dE_1)^n} = \tfrac{1}{2}G(-1)^n n!\, 2^n \frac{1}{(1+2E_1)^{n+1}}, \quad n \geqslant 2. \tag{38}$$

It follows now from (38), (34) and (14) that the order of $\mathbf{W}_{(n)}$ can be estimated by

$$W_{(0)} = G0(\gamma^2), \quad \|\mathbf{W}_{(1)}\| = G0(\gamma), \quad \|\mathbf{W}_{(n)}\| = Gn!\,2^{n-1}[1+0(\gamma)] = G0(1), \quad n \geqslant 1. \tag{39}$$

We have applied the same technique to several material laws other than (35) and in each case the estimates were of the order of (39), which indicates that (39) does not depend upon the material law. Indeed, for $\gamma \ll 1$ and with the absence of initial stresses the function W of any elastic material can be approximately described by a quadratic function of strains, which immediately leads to the order estimates $(39)_{1,2}$ and $(39)_3$ for $n = 2$.

It follows now from (39) and (29) that, to within the relative error $0(\theta^2)$ in the bending energy, the shell strain energy function (33) can be approximated by

$$\Phi = hW_{(0)}[1+0(\theta^2)] + \frac{h^3}{12}\mathbf{W}_{(1)}\cdot(\pi - b_\lambda^\lambda\chi)[1+0(\theta^2)] + \frac{h^3}{24}\mathbf{W}_{(2)}\cdot(\chi\otimes\chi)[1+0(\theta^2)]. \tag{40}$$

We call the expression (40) *the consistent first approximation to the elastic strain energy of rubber-like shells.*

The estimate in (40) suggests that within the same relative error it is possible to simplify consistently the strain measures (25) and (26) for various ranges of admissible strains. Note that within the admissible order of at most moderate bending strains $h\kappa \leqslant 0(\theta)$ three principal cases can be discussed:

(A) $\gamma = 0(1)$, i.e. $h\kappa/\gamma = 0(\theta)$. Here $W_{(0)} = G0(1)$, $\|\mathbf{W}_{(1)}\| = G0(1)$, $\|\mathbf{W}_{(2)}\| = G0(1)$. Since $h\|\chi\| = 0(\theta)$ and $h^2\|\pi\| = 0(\theta^2)$ the strain energy (40) is dominated, to within the relative error $0(\theta^2)$, by the first term, which leads to

$$\Phi = hW_{(0)}[1+0(\theta^2)]. \tag{41}$$

The strain energy function (41) describes the membrane theory of rubber-like shells. The possible simplifications of χ and π are irrelevant here, since those strain measures themselves are included in the error margin of (41).

(B) $\gamma = 0(\theta^2)$, i.e. $\gamma/h\kappa = 0(\theta)$. In this case $W_{(0)} = G0(\theta^4)$, $\|\mathbf{W}_{(1)}\| = G0(\theta^2)$, $\|\mathbf{W}_{(2)}\| = G0(1)$. Since again $h\|\chi\| = 0(\theta)$ and $h^2\|\pi\| = 0(\theta^2)$, the strain energy (40) is dominated, to within the relative error $0(\theta^2)$, by the last term which leads to

$$\Phi = \frac{h^3}{24}\mathbf{W}_{(2)}\cdot(\chi\otimes\chi)[1+0(\theta^2)]. \tag{42}$$

Within the same relative error $0(\theta^2)$ the change of curvature measure χ in (42) can be simplified to be

$$\chi_{\alpha\beta} = \kappa_{\alpha\beta} + 0(\theta^3/h). \tag{43}$$

The possible simplification of π is again irrelevant here, since π itself falls into the error margin of (42). Additionally, here $\mathbf{W}_{(2)}(\gamma)$ can be expanded into series of γ, and to within the relative error $0(\theta^2)$ only the first constant term for $\gamma = 0$ can be taken into account. This leads to the modified elasticity tensor $\mathbf{H}$ defined in the reference configuration. It corresponds to the classical one of the geometrically non-linear theory of elastic shells

$$\mathbf{W}_{(2)}(\gamma) = \mathbf{W}_{(2)}(\mathbf{0})[1+0(\theta^2)]$$
$$= \left.\frac{\partial^2 W_{(0)}}{\partial\gamma\,\partial\gamma}\right|_{\gamma=0}[1+0(\theta^2)] = \mathbf{H}[1+0(\theta^2)], \tag{44}$$

$$H^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\left(a^{\alpha\lambda}a^{\beta\mu}+a^{\alpha\mu}a^{\beta\lambda}+\frac{2\nu}{1-\nu}\,a^{\alpha\beta}a^{\lambda\mu}\right), \tag{45}$$

where E is Young's modulus of the material in the undeformed configuration and ν is the Poisson's ratio which, for the incompressible material, should be taken to be $\nu = 0.5$.

The strain energy function (42), with (43) and (44), describes the inextensional bending theory of rubber-like shells.

(C) $\gamma = 0(\theta)$, i.e. $\gamma/h\kappa = 0(1)$. In this case $W_{(0)} = G0(\theta^2)$, $\|\mathbf{W}_{(1)}\| = G0(\theta)$, $\|\mathbf{W}_{(2)}\| = G0(1)$, which together with $h\|\chi\| = 0(\theta)$ and $h^2\|\pi\| = 0(\theta^2)$ shows that the first and the third term of (40) are $Gh\,0(\theta^2)$, while the second term is $Gh\,0(\theta^3)$. Therefore, to within the relative error $0(\theta^2)$ all terms in (40) should be taken into account. However, the strain measures appearing in the second term of (40) can be approximated to within the relative error $0(\theta)$ by

$$2\pi_{\alpha\beta} = \kappa_\alpha^\lambda\kappa_{\lambda\beta}+(b_{\alpha\beta}-\kappa_{\alpha\beta})\kappa_\lambda^\lambda+b_\alpha^\lambda\kappa_{\lambda\beta}+b_\beta^\lambda\kappa_{\alpha\lambda}+0(\theta^3/h^2),$$
$$b_\lambda^\lambda\chi_{\alpha\beta} = b_\lambda^\lambda\kappa_{\alpha\beta}+0(\theta^3/h^2). \tag{46}$$

As a result, the strain energy function (40) with (25) and (46) describes the bending theory of rubber-like shells.

Similar discussion performed for the weak bending case, when $\gamma = 0(\sqrt{\theta})$, $h\kappa/\gamma = 0(\sqrt{\theta})$, and for the strong bending case, when $\gamma = 0(\theta\sqrt{\theta})$, $\gamma/h\kappa = 0(\sqrt{\theta})$, leads to within the relative error $0(\theta^2)$ essentially to the results given for the case C of the bending theory of rubber-like shells.

As a result of the discussion given above, within the whole range of admissible membrane strains and curvature changes (14) and within the associated range of thickness-to-radii ratios (13), the consistent first approximation to the elastic strain energy of the rubber-like shell takes the form (40), where the full expression (25) for χ and the consistently approximated expressions (46) for the corresponding strain measures appearing in the second term can be used.

Let us note that the maximal influence of the second term in (40) has been found to be in the case C of the bending theory, where the relative error of the strain energy function resulting from dropping those terms can be estimated to be $\frac{1}{12}0(\theta)$ with respect to the first term describing the membrane strain energy. Although such an error cannot be included in the consistent relative error $0(\theta^2)$, it is apparent that the contribution of the second term to the strain energy function (40) is very small indeed, considerably reduced due to the additional multiplier 1/12 to the terms, whose contribution to (40) has already been estimated to be at most $0(\theta)$ as compared with the first leading term. Therefore, in many engineering calculations of rubber-like shell problems it is reasonable to simplify (40) further by dropping the second term, which leads to

$$\Phi \cong hW_{(0)}+\frac{h^3}{24}\mathbf{W}_{(2)}\cdot(\chi\otimes\chi). \tag{47}$$

We call the form (47) *the simplest approximation to the elastic strain energy of rubber-like* *shells*. It takes into account only the primary parts of the shell strain energy due to the stretching and bending of its reference surface and due to the corresponding transverse normal strains.

Our numerical tests show that the use of the simplest approximation (47) needs over 33% less computer time than the consistent first approximation (40). At the same time, in most of the examples the numerical results are essentially the same, while for various examples the differences in the numerical results do not exceed 2%. It seems, therefore, that for many engineering applications the simplest approximation (47) might be a very attractive alternative to (40).

6. CONSTITUTIVE EQUATIONS

If the strain measures γ, χ and π are treated as independent, the first variation of (40) gives

$$\delta\Phi = n^{\alpha\beta}\delta\gamma_{\alpha\beta} + m^{\alpha\beta}\delta\chi_{\alpha\beta} + \underline{k^{\alpha\beta}\delta\pi_{\alpha\beta}} \tag{48}$$

with the constitutive equations :

$$\begin{aligned} n^{\alpha\beta} &= \frac{\partial\Phi}{\partial\gamma_{\alpha\beta}} = hG^{\alpha\beta} + \frac{h^3}{24}\left[2G^{\alpha\beta\lambda\mu}(\pi_{\lambda\mu} - b^{\sigma}_{\lambda}\chi_{\sigma\mu}) + G^{\alpha\beta\lambda\mu\kappa\rho}\chi_{\lambda\mu}\chi_{\kappa\rho}\right], \\ m^{\alpha\beta} &= \frac{\partial\Phi}{\partial\chi_{\alpha\beta}} = \frac{h^3}{12}\left(-G^{\sigma\beta}b^{\alpha}_{\sigma} + G^{\alpha\beta\lambda\mu}\chi_{\lambda\mu}\right), \\ k^{\alpha\beta} &= \frac{\partial\Phi}{\partial\pi_{\alpha\beta}} = \frac{h^3}{12}G^{\alpha\beta}. \end{aligned} \tag{49}$$

The components of tangent elasticity tensors of various orders are defined analogously to (34) by

$$G^{\alpha\beta} = \frac{\partial W_{(0)}}{\partial\gamma_{\alpha\beta}}, \quad G^{\alpha\beta\kappa\rho} = \frac{\partial^2 W_{(0)}}{\partial\gamma_{\alpha\beta}\,\partial\gamma_{\kappa\rho}}, \quad G^{\alpha\beta\kappa\rho\lambda\mu} = \frac{\partial^3 W_{(0)}}{\partial\gamma_{\alpha\beta}\,\partial\gamma_{\kappa\rho}\,\partial\gamma_{\lambda\mu}}. \tag{50}$$

In (49), $n^{\alpha\beta}$, $m^{\alpha\beta}$ and $k^{\alpha\beta}$ are the symmetric surface stress measures which are work-conjugate to the strain measures $\gamma_{\alpha\beta}$, $\chi_{\alpha\beta}$ and $\pi_{\alpha\beta}$. If the simplest approximation (47) is used, the terms underlined by the solid line in (48) and (49) have to be omitted.

If $\kappa_{\alpha\beta}$ is used as a bending strain measure, it follows from (25)–(28) that

$$\begin{aligned} \delta\chi_{\alpha\beta} &= \frac{\partial\chi_{\alpha\beta}}{\partial\gamma_{\lambda\mu}}\delta\gamma_{\lambda\mu} + \frac{\partial\chi_{\alpha\beta}}{\partial\kappa_{\lambda\mu}}\delta\kappa_{\lambda\mu}, \\ \delta\pi_{\alpha\beta} &= \frac{\partial\pi_{\alpha\beta}}{\partial\gamma_{\lambda\mu}}\delta\gamma_{\lambda\mu} + \frac{\partial\pi_{\alpha\beta}}{\partial\kappa_{\lambda\mu}}\delta\kappa_{\lambda\mu}. \end{aligned} \tag{51}$$

This allows the expression of the variation of Φ in a form which is well known from the classical geometrically non-linear theory of thin elastic shells

$$\delta\Phi = N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta}, \tag{52}$$

only now we have to introduce

$$N^{\alpha\beta} = n^{\alpha\beta} + m^{\lambda\mu}\frac{\partial \chi_{\lambda\mu}}{\partial \gamma_{\alpha\beta}} + k^{\lambda\mu}\frac{\partial \pi_{\lambda\mu}}{\partial \gamma_{\alpha\beta}},$$

$$M^{\alpha\beta} = m^{\lambda\mu}\frac{\partial \chi_{\lambda\mu}}{\partial \kappa_{\alpha\beta}} + k^{\lambda\mu}\frac{\partial \pi_{\lambda\mu}}{\partial \kappa_{\alpha\beta}}. \tag{53}$$

The symmetric surface stress measures $N^{\alpha\beta}$, $M^{\alpha\beta}$ are work-conjugate to the surface strain measures $\gamma_{\alpha\beta}$, $\kappa_{\alpha\beta}$, respectively.

Within the consistent first approximation it follows from (25) and (46) that

$$\delta\chi_{\alpha\beta} = \lambda^3(\underline{b_{\alpha\beta}} - \kappa_{\alpha\beta})[(1+2\gamma^\kappa_\kappa)a^{\lambda\mu} - 2\gamma^{\lambda\mu}]\delta\gamma_{\lambda\mu} + \lambda\delta\kappa_{\alpha\beta} + \underline{b^\lambda_\alpha\delta\gamma_{\lambda\beta} + b^\lambda_\beta\delta\gamma_{\alpha\lambda}},$$

$$2\delta\pi_{\alpha\beta} = \kappa^\lambda_\alpha\delta\kappa_{\lambda\beta} + \kappa^\lambda_\beta\delta\kappa_{\alpha\lambda} + (\underline{b_{\alpha\beta}} - \kappa_{\alpha\beta})a^{\lambda\mu}\delta\kappa_{\lambda\mu} - \kappa^\lambda_\lambda\delta\kappa_{\alpha\beta} + \underline{b^\lambda_\alpha\delta\kappa_{\lambda\beta} + b^\lambda_\beta\delta\kappa_{\alpha\lambda}},$$

$$b^\lambda_\lambda\delta\chi_{\alpha\beta} \approx \underline{b^\lambda_\lambda\delta\kappa_{\alpha\beta}} \quad \text{(see also (46)}_2\text{)}. \tag{54}$$

This allows us to give the constitutive equations for $N^{\alpha\beta}$ and $M^{\alpha\beta}$ in the following explicit form:

$$N^{\alpha\beta} = hG^{\alpha\beta} + \frac{h^3}{12}\Big\{G^{\alpha\beta\lambda\mu}(\pi_{\lambda\mu} - \underline{b^\kappa_\kappa\kappa_{\lambda\mu}}) - G^{\gamma\rho}b^\sigma_\sigma\lambda^3(b_{\gamma\rho} - \kappa_{\gamma\rho})[(1+2\gamma^\kappa_\kappa)a^{\alpha\beta} - 2\gamma^{\alpha\beta}]$$

$$- G^{\kappa\beta}b^\sigma_\sigma b^\alpha_\kappa - G^{\kappa\alpha}b^\sigma_\sigma b^\beta_\kappa + G^{\lambda\mu\gamma\rho}\lambda^3\chi_{\lambda\mu}(\underline{b_{\gamma\rho}} - \kappa_{\gamma\rho})[(1+2\gamma^\kappa_\kappa)a^{\alpha\beta} - 2\gamma^{\alpha\beta}]$$

$$+ (\underline{G^{\kappa\beta\lambda\mu}b^\alpha_\kappa + G^{\alpha\kappa\lambda\mu}b^\beta_\kappa})\chi_{\lambda\mu} + \tfrac{1}{2}G^{\alpha\beta\lambda\mu\gamma\rho}\chi_{\lambda\mu}\chi_{\gamma\rho}\Big\},$$

$$M^{\alpha\beta} = \frac{h^3}{24}[\underline{G^{\alpha\kappa}(b^\beta_\kappa + \kappa^\beta_\kappa) + G^{\kappa\beta}(b^\alpha_\kappa + \kappa^\alpha_\kappa) + G^{\lambda\mu}(b_{\lambda\mu} - \kappa_{\lambda\mu})a^{\alpha\beta} - G^{\alpha\beta}\kappa^\lambda_\lambda}$$

$$- 2G^{\alpha\beta}b^\sigma_\sigma\lambda + 2G^{\alpha\beta\lambda\mu}\lambda\chi_{\lambda\mu}]. \tag{55}$$

Again, if the simplest approximation (47) is used the terms underlined by the solid line in (55) have to be omitted.

When the initial thickness of the shell is assumed to be small, i.e. $h/R = 0(\theta^2)$, the two last terms in (25), the term with the dashed line in (40) and all terms including $b_{\alpha\beta}$ in (46) can be omitted within the error of the consistent first approximation, and $\chi_{\alpha\beta}$ in (25) can be approximated by $\lambda\kappa_{\alpha\beta}$. In this case terms underlined by the dashed line will not appear in the corresponding relations (48), (49), (54) and (55) and therefore our constitutive equations become independent of the initial curvature of the shell reference surface, as was already suggested by Simmonds (1985). However, the structure of our so reduced constitutive equations (55) seems to be simpler than the one following from Simmonds (1985).

7. LAGRANGIAN DISPLACEMENT FORM OF SHELL EQUATIONS

The surface strain measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ are known functions of the displacement field $\mathbf{u} = u_\alpha\mathbf{a}^\alpha + w\mathbf{n}$. They are given by the following exact vector expressions [cf. Pietraszkiewicz (1983)]

$$\gamma_{\alpha\beta} = \tfrac{1}{2}(\bar{\mathbf{a}}_\alpha\cdot\bar{\mathbf{a}}_\beta - a_{\alpha\beta}), \quad \kappa_{\alpha\beta} = \bar{\mathbf{a}}_\alpha\cdot\bar{\mathbf{n}}_{,\beta} + b_{\alpha\beta}, \tag{56}$$

where

$$\bar{\mathbf{a}}_\alpha = \mathbf{a}_\alpha + \mathbf{u}_{,\alpha}, \quad \bar{\mathbf{n}} = \tfrac{1}{2}\lambda\varepsilon^{\alpha\beta}\bar{\mathbf{a}}_\alpha \times \bar{\mathbf{a}}_\beta, \quad \lambda^{-2} = \frac{\bar{a}}{a} = \tfrac{1}{2}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}(\bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}_\beta)(\bar{\mathbf{a}}_\lambda \cdot \bar{\mathbf{a}}_\mu). \tag{57}$$

Let the boundary $\mathfrak{C}$ of $\mathfrak{M}$ consist of a finite set of piecewise smooth curves with the position vector $\mathbf{r}(s) = \mathbf{r}[\theta^\alpha(s)]$, where s is the arc length parameter along $\mathfrak{C}$. At each regular point $M \in \mathfrak{C}$ we introduce the unit tangent vector $\mathbf{t} = \mathrm{d}\mathbf{r}/\mathrm{d}s \equiv \mathbf{r}' = t^\alpha \mathbf{a}_\alpha$ and the outward unit normal vector $\boldsymbol{\nu} = \mathbf{r}_{,\nu} = \nu^\alpha \mathbf{a}_\alpha = \mathbf{t} \times \mathbf{n}$, where $(\ )_{,\nu}$ denotes the outward normal derivative at $\mathfrak{C}$.

We describe the deformation of the shell boundary by the displacement field $\mathbf{u}(s)$ and its normal derivative $\mathbf{u}_{,\nu}(s)$. Then the geometry of the deformed boundary $\bar{\mathfrak{C}}$ can be expressed by:

$$\bar{\mathbf{r}}' = \bar{\mathbf{a}}_\alpha t^\alpha = \mathbf{t} + \mathbf{u}', \quad \bar{\mathbf{r}}_{,\nu} = \bar{\mathbf{a}}_\alpha \nu^\alpha = \boldsymbol{\nu} + \mathbf{u}_{,\nu}, \quad \bar{\mathbf{n}} = \lambda \bar{\mathbf{r}}_{,\nu} \times \bar{\mathbf{r}}', \quad \lambda^{-1} = |\bar{\mathbf{r}}_{,\nu} \times \bar{\mathbf{r}}'|. \tag{58}$$

Let $\mathbf{p} = p^\alpha \mathbf{a}_\alpha + p\mathbf{n}$ be the external surface force resultant per unit area of $\mathfrak{M}$ and let the undeformed boundary $\mathfrak{C}$ consist of two parts, $\mathfrak{C}_u$ and $\mathfrak{C}_f$, with $\mathfrak{C}_u \cup \mathfrak{C}_f = \mathfrak{C}$, where on $\mathfrak{C}_u$ geometric quantities and on $\mathfrak{C}_f$ static quantities are prescribed. We assume that on $\mathfrak{C}_f$ the external boundary force resultant $\mathbf{T}^* = T^*_\nu \boldsymbol{\nu} + T^*_t \mathbf{t} + T^* \mathbf{n}$ and the moment resultant $\mathbf{H}^* = H^*_\nu \boldsymbol{\nu} + H^*_t \mathbf{t} + H^* \mathbf{n}$ are given, both per unit length of $\mathfrak{C}$. They are determined through the prescribed external boundary surface force vector $\mathbf{f}^*$, defined per unit area of the lateral boundary surface $\partial\mathfrak{P}$ of the undeformed shell

$$\mathbf{T}^* \,\mathrm{d}s = \int_{-h/2}^{+h/2} \mathbf{f}^* \,\mathrm{d}A, \quad \mathbf{H}^* \,\mathrm{d}s = \int_{-h/2}^{+h/2} \mathbf{f}^* \zeta \,\mathrm{d}A. \tag{59}$$

If geometric and static quantities are given on the same part of $\mathfrak{C}$ they must be complementary with respect to the external virtual work (63).

Let us denote displacement fields admissible, if they satisfy sufficient differentiability conditions in $\mathfrak{M}$ and on $\mathfrak{C}_u$ the geometric boundary conditions (74) in homogeneous form. Then the deformed shell is in an equilibrium state if for all geometrically admissible virtual displacement fields δu the following Lagrangean principle of virtual displacements is satisfied:

$$G[\mathbf{u}; \delta\mathbf{u}] = 0, \tag{60}$$

where the functional of virtual work G is defined through the internal virtual work G_i and the external virtual work G_e by:

$$G = G_i - G_e, \tag{61}$$

$$G_i = \iint_{\mathfrak{M}} (N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta}) \,\mathrm{d}A \tag{62}$$

$$G_e = \iint_{\mathfrak{M}} \mathbf{p} \cdot \delta\mathbf{u} \,\mathrm{d}A + \int_{\mathfrak{C}_f} (\mathbf{T}^* \cdot \delta\mathbf{u} + \mathbf{H}^* \cdot \delta\bar{\mathbf{n}}) \,\mathrm{d}s. \tag{63}$$

From $(58)_3$ it follows that on $\mathfrak{C}$ the normal vector $\bar{\mathbf{n}}(s)$ depends on the displacement derivatives $\mathbf{u}'(s)$ and $\mathbf{u}_{,\nu}(s)$ and therefore can be given as a function of the two vector fields $\mathbf{u}(s)$ and $\mathbf{u}_{,\nu}(s)$ with six independent components. Introducing in $\mathfrak{M}$ and on $\mathfrak{C}$ the constraints

$$\bar{\mathbf{n}} \cdot \bar{\mathbf{a}}_\alpha = 0, \quad \bar{\mathbf{n}} \cdot \bar{\mathbf{n}} = 1 \quad \text{in } \mathfrak{M} \tag{64}$$

$$\bar{\mathbf{n}} \cdot \bar{\mathbf{r}}' = 0, \quad \bar{\mathbf{n}} \cdot \bar{\mathbf{n}} = 1 \quad \text{on } \mathfrak{C} \tag{65}$$

then the set of independent geometrical variables on $\mathfrak{C}$ is reduced to four. In this paper we choose as geometrical variables on $\mathfrak{C}$ the displacement field $\mathbf{u}(s)$ and the scalar component $n_v(s)$ of the normal vector $\bar{\mathbf{n}} = n_v \mathbf{v} + n_t \mathbf{t} + n\mathbf{n}$.

The variation of $\bar{\mathbf{n}}$ in $\mathfrak{M}$ and on $\mathfrak{C}$ can be found by varying the constraints (64) and (65) leading to

$$\delta\bar{\mathbf{n}} = -\bar{\mathbf{a}}^{\beta}(\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\beta}) \quad \text{in } \mathfrak{M}, \tag{66}$$

$$\delta\bar{\mathbf{n}} = \frac{1}{a_v}[(\mathbf{v}\times\bar{\mathbf{n}})\bar{\mathbf{n}}\cdot\delta\mathbf{u}' + (\bar{\mathbf{r}}'\times\bar{\mathbf{n}})\delta n_v],$$

$$a_v = (\bar{\mathbf{r}}'\times\bar{\mathbf{n}})\cdot\mathbf{v}, \quad \text{on } \mathfrak{C} \tag{67}$$

where the scalar function $n_v(s)$

$$n_v = \bar{\mathbf{n}}\cdot\mathbf{v} = \lambda(\mathbf{u}'\times\mathbf{v}-\mathbf{n})\cdot\mathbf{u}_{,v} \tag{68}$$

was first introduced by Pietraszkiewicz and Szwabowicz (1981). The use of scalar functions other than n_v as fourth geometrical boundary variable is considered in Makowski and Pietraszkiewicz (1989).

Using (64) the variation of the strain measures (56) leads to:

$$\begin{aligned} \delta\gamma_{\alpha\beta} &= \tfrac{1}{2}(\bar{\mathbf{a}}_{\alpha}\cdot\delta\mathbf{u}_{,\beta}+\bar{\mathbf{a}}_{\beta}\cdot\delta\mathbf{u}_{,\alpha}) \\ \delta\kappa_{\alpha\beta} &= \tfrac{1}{2}(\bar{\mathbf{n}}_{,\alpha}\cdot\delta\mathbf{u}_{,\beta}+\bar{\mathbf{n}}_{,\beta}\cdot\delta\mathbf{u}_{,\alpha}+\bar{\mathbf{a}}_{\alpha}\cdot\delta\bar{\mathbf{n}}_{,\beta}+\bar{\mathbf{a}}_{\beta}\cdot\delta\bar{\mathbf{n}}_{,\alpha}) \end{aligned} \tag{69}$$

with (66)–(69) the virtual work functional (61)–(63) can be transformed into

$$G[\mathbf{u};\delta\mathbf{u}] = -\iint_{\mathfrak{M}}(\mathbf{T}^{\beta}|_{\beta}+\mathbf{p})\cdot\delta\mathbf{u}+\int_{\mathfrak{C}_f}[(\mathbf{P}-\mathbf{P}^*)\cdot\delta\mathbf{u}+(M-M^*)\delta n_v]\,\mathrm{d}s$$
$$+\sum_j(\mathbf{F}_j-\mathbf{F}_j^*)\cdot\delta\mathbf{u}_j+\int_{\mathfrak{C}_u}(\mathbf{P}\cdot\delta\mathbf{u}+M\delta n_v)\,\mathrm{d}s, \tag{70}$$

where asterisks indicate prescribed boundary functions, but depending on the deformation, and the sum $\sum_j$ is taken over all corner points $M_j\in\mathfrak{C}_f$. In (70) the following short notations are used:

$$\begin{aligned} \mathbf{T}^{\beta} &= N^{\alpha\beta}\bar{\mathbf{a}}_{\alpha}+M^{\alpha\beta}\bar{\mathbf{n}}_{,\alpha}+[(M^{\lambda\alpha}\bar{\mathbf{a}}_{\lambda})|_{\alpha}\cdot\bar{\mathbf{a}}^{\beta}]\bar{\mathbf{n}}, \\ \mathbf{P} &= \mathbf{T}^{\beta}v_{\beta}+\mathbf{F}', \quad \mathbf{P}^* = \mathbf{T}^*+\mathbf{F}^{*\prime}, \\ \mathbf{F} &= -\frac{1}{a_v}[(\bar{\mathbf{n}}\times\bar{\mathbf{a}}_{\alpha})\cdot\mathbf{v}]M^{\alpha\beta}v_{\alpha}\bar{\mathbf{n}}, \quad \mathbf{F}^* = -\frac{1}{a_v}[(\bar{\mathbf{n}}\times\mathbf{H}^*)\cdot\mathbf{v}]\bar{\mathbf{n}}, \\ M &= \frac{1}{a_v}(\bar{\mathbf{n}}\times\bar{\mathbf{a}}_{\alpha})\cdot\bar{\mathbf{r}}'M^{\alpha\beta}v_{\beta}, \quad M^* = \frac{1}{a_v}(\bar{\mathbf{n}}\times\mathbf{H}^*)\cdot\bar{\mathbf{r}}', \\ \mathbf{F}_j &= \mathbf{F}(s_j+0)-\mathbf{F}(s_j-0), \quad \mathbf{F}_j^* = \mathbf{F}^*(s_j+0)-\mathbf{F}^*(s_j-0). \end{aligned} \tag{71}$$

It follows from (70) that on $\mathfrak{C}$ the static quantities $\mathbf{P}(s) = \mathbf{P}[\mathbf{u}(s), n_v(s)]$ and $M(s) = M[\mathbf{u}(s), n_v(s)]$ are work-conjugate to the chosen geometric quantities $\mathbf{u}(s)$ and $n_v(s)$, respectively.

Introducing (71) into the virtual work principle (60) we obtain the Lagrangian equilibrium equations

$$\mathbf{T}^{\beta}|_{\beta} + \mathbf{p} = 0 \quad \text{in } \mathfrak{M} \tag{72}$$

and the static boundary and corner conditions

$$\mathbf{P}(s) = \mathbf{P}^*(s), \quad M(s) = M^*(s) \quad \text{on } \mathfrak{C}_f$$
$$\mathbf{F}_i = \mathbf{F}_i \quad \text{at each corner point } M_i \in \mathfrak{C}_f. \tag{73}$$

The corresponding work-conjugate geometric boundary and corner conditions are:

$$\left.\begin{aligned} \mathbf{u}(s) &= \mathbf{u}^*(s), \\ n_\nu(\mathbf{u}(s), \mathbf{u}_{,\nu}(s)) &= n_\nu^*(s) \end{aligned}\right\} \text{on } \mathfrak{C}_u$$
$$\mathbf{u}_k = \mathbf{u}_k^* \quad \text{at each corner point } M_k \in \mathfrak{C}_u. \tag{74}$$

With (74) the last integral term of (70) vanishes.

It should be pointed out that the equilibrium equations (72) and the static boundary and corner conditions (73) are linear in $N^{\alpha\beta}$ and $M^{\alpha\beta}$, but non-rational in the displacement $\mathbf{u}$. This non-rationality is caused by the presence of λ in the definition of $\kappa_{\alpha\beta}$ in $\mathfrak{M}$ and by the non-rationality of $\bar{\mathbf{n}}$ on $\mathfrak{C}$. The geometric boundary condition $(74)_2$ depends non-linearly on the boundary displacement $\mathbf{u}(s)$ and its normal derivative $\mathbf{u}_{,\nu}(s)$.

In the case when the external loadings acting on the shell have a potential $V(\mathbf{u})$ such that $G_e[\mathbf{u}; \delta\mathbf{u}] = -\delta V[\mathbf{u}; \delta\mathbf{u}]$ a functional of total potential energy exists defined by

$$J(\mathbf{u}) = \iint_{\mathfrak{M}} \Phi[\gamma_{\alpha\beta}(\mathbf{u}), \kappa_{\alpha\beta}(\mathbf{u})]\, \mathrm{d}A + V(\mathbf{u}), \tag{75}$$

where for the strain energy density $\Phi(\gamma_{\alpha\beta}, \kappa_{\alpha\beta})$ we have to introduce (40) or (47) expressed as a function of the displacement $\mathbf{u}$. Then we can formulate the principle of total potential energy for large strain shell problems as follows: If for all geometrically admissible displacement fields $\mathbf{u}$ satisfying the geometric boundary conditions (74) the variation δJ vanishes

$$\delta J(\mathbf{u}; \delta\mathbf{u}) = 0 \tag{76}$$

then $\mathbf{u}$ is an equilibrium configuration satisfying (72) and (73) as Euler–Lagrange equations.

We want to point out that (76) is a stationary variational principle. Following Stumpf (1976, 1979) we can also consider under which additional restrictions it is a minimum principle and a dual maximum principle can be formulated. Other variational principles can be constructed for the presented large strain shell theory following Stumpf (1976), Schmidt and Pietraszkiewicz (1981), Schmidt (1986) and Szwabowicz (1986).

8. INCREMENTAL FORMULATION

Let us assume that the external loads are specified by a single parameter $\alpha \in A \subset R$. For smoothly varying α the regular solutions of the shell boundary value problem (72)–(74) form an equilibrium path $\mathbf{u}(\alpha)$. By virtue of (60) $\mathbf{u}(\alpha)$ is a weak solution of the boundary value problem for a fixed α if $G[\mathbf{u}(\alpha); \delta\mathbf{u}] = 0$ for all kinematically admissible virtual displacements $\delta\mathbf{u}$. In the numerical approach the unknown equilibrium path $\mathbf{u}(\alpha)$ is divided into a finite number of equilibrium states corresponding to discrete values of $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_N \in A$ of the load parameter.

Let $\mathbf{u}_n = \mathbf{u}(\alpha_n)$ be the equilibrium solution we are looking for, and let $\mathbf{u}_n^{(i)}$ denote the ith approximation to $\mathbf{u}_n$, which may not belong to the equilibrium path. In order to construct a correction $\Delta\mathbf{u}_n^{(i+1)}$ allowing us to determine the next approximation $\mathbf{u}_n^{(i+1)} = \mathbf{u}_n^{(i)} + \Delta\mathbf{u}_n^{(i+1)}$ to $\mathbf{u}_n$, we use the consistent linearization of $G[\mathbf{u}; \delta\mathbf{u}]$ at the approximation $\mathbf{u}_n^{(i)}$, which allows us to replace eqn (60) by

$$\Delta G[\mathbf{u}_n^{(i)}; \Delta\mathbf{u}_n^{(i+1)}, \delta\mathbf{u}] + G[\mathbf{u}_n^{(i)}; \delta\mathbf{u}] = 0. \tag{77}$$

The first term in (77) denotes the directional Gâteaux derivative of the functional (61) calculated at the point $\mathbf{u}_n^{(i)}$ in the kinematically admissible direction $\Delta\mathbf{u}_n^{(i+1)}$. This term being linear in the unknown $\Delta\mathbf{u}_n^{(i+1)}$ leads to the definition of the stiffness matrix of the problem. If $\mathbf{u}_n^{(i)}$ does not belong to the equilibrium path, the second term in (77) allows the calculation of the unbalanced force vector.

In order to simplify the notation, in the following part of this chapter we set $\mathbf{u}_n^{(i)} \equiv \mathbf{u}$, $\Delta\mathbf{u}_n^{(i+1)} \equiv \Delta\mathbf{u}$, while the values of corresponding external loads at $\mathbf{u}_n$ will be denoted by $\mathbf{p}(\alpha_n) = \mathbf{p}_n \equiv \mathbf{p}$, $\mathbf{T}(\alpha_n) = \mathbf{T}_n \equiv \mathbf{T}$, $\mathbf{H}(\alpha_n) = \mathbf{H}_n \equiv \mathbf{H}$.

Let us consider a curve $\mathbf{u}(\eta)$ through the ith approximation to $\mathbf{u}_n$ such that in the neighbourhood of $\mathbf{u}$

$$\mathbf{u}(\eta) = \mathbf{u} + \eta\Delta\mathbf{u}. \tag{78}$$

Then the Gâteaux derivative of the functional G taken at $\mathbf{u}$ in the kinematically admissible direction $\Delta\mathbf{u}$ is given by

$$\Delta G[\mathbf{u}; \Delta\mathbf{u}, \delta\mathbf{u}] = \frac{\mathrm{d}}{\mathrm{d}\eta} G[\mathbf{u}(\eta); \delta\mathbf{u}]|_{\eta=0}, \tag{79}$$

where $G[\mathbf{u}(\eta); \delta\mathbf{u}]$ is defined by (61)–(63) with $\mathbf{u}$ being replaced by $\mathbf{u}(\eta)$. Along the curve $\mathbf{u}(\eta)$ the corresponding Lagrangian surface stress and strain measures in $\mathfrak{M}$ are denoted by $\mathbf{N}(\eta)$, $\mathbf{M}(\eta)$, $\gamma(\eta)$, $\boldsymbol{\kappa}(\eta)$, while the corresponding Lagrangian surface forces in $\mathfrak{M}$ and boundary forces and moments on $\mathfrak{C}$ are denoted by $\mathbf{p}(\eta)$, $\mathbf{T}(\eta)$ and $\mathbf{H}(\eta)$, respectively.

In the internal part of $\mathfrak{M}$ the linearized increments of $\bar{\mathbf{a}}^\beta$, $\bar{\mathbf{n}}$ and $\delta\bar{\mathbf{n}}$ at $\mathbf{u}$ in the direction of $\Delta\mathbf{u}$ follow from the identities $\bar{\mathbf{a}}^\beta\cdot\bar{\mathbf{a}}_\alpha = \delta^\beta_\alpha$, $\bar{\mathbf{a}}^\beta\cdot\bar{\mathbf{n}} = 0$ and (66):

$$\begin{aligned}\Delta\bar{\mathbf{a}}^\beta &= -(\bar{\mathbf{a}}^\beta\cdot\Delta\mathbf{u}_{,\kappa})\bar{\mathbf{a}}^\kappa + \bar{a}^{\beta\kappa}\Delta\mathbf{u}_{,\kappa},\\ \Delta\bar{\mathbf{n}} &= -\bar{\mathbf{a}}^\kappa(\bar{\mathbf{n}}\cdot\Delta\mathbf{u}_{,\kappa}),\end{aligned} \tag{80}$$

$$\begin{aligned}\Delta(\delta\bar{\mathbf{n}}) &= \mathbf{B}^\beta\cdot\delta\mathbf{u}_{,\beta},\\ \mathbf{B} &= [(\mathbf{a}\cdot\Delta\mathbf{u}_{,\kappa})\mathbf{a} - a\Delta\mathbf{u}_{,\kappa}]\otimes\mathbf{n} + (\mathbf{n}\cdot\Delta\mathbf{u}_{,\lambda})\mathbf{a}\otimes\mathbf{a}.\end{aligned} \tag{81}$$

This leads to the following incremental strain measures and their incremental variations:

$$\begin{aligned}\Delta\gamma_{\alpha\beta} &= \tfrac{1}{2}(\bar{\mathbf{a}}_\alpha\cdot\Delta\mathbf{u}_{,\beta} + \bar{\mathbf{a}}_\beta\cdot\Delta\mathbf{u}_{,\alpha}),\\ \Delta\kappa_{\alpha\beta} &= \tfrac{1}{2}\{\bar{\mathbf{n}}_{,\alpha}\cdot\Delta\mathbf{u}_{,\beta} + \bar{\mathbf{n}}_{,\beta}\cdot\Delta\mathbf{u}_{,\alpha} + \bar{\mathbf{a}}_\alpha\cdot\Delta\bar{\mathbf{n}}_{,\beta} + \bar{\mathbf{a}}_{,\beta}\cdot\Delta\bar{\mathbf{n}}_{,\alpha}\},\end{aligned} \tag{82}$$

$$\begin{aligned}\Delta(\delta\gamma_{\alpha\beta}) &= \tfrac{1}{2}(\Delta\mathbf{u}_{,\alpha}\cdot\delta\mathbf{u}_{,\beta} + \Delta\mathbf{u}_{,\beta}\cdot\delta\mathbf{u}_{,\alpha}),\\ \Delta(\delta\kappa_{\alpha\beta}) &= \tfrac{1}{2}\{\Delta\bar{\mathbf{n}}_{,\alpha}\cdot\delta\mathbf{u}_{,\beta} + \Delta\bar{\mathbf{n}}_{,\beta}\cdot\delta\mathbf{u}_{,\alpha} + \Delta\mathbf{u}_{,\alpha}\cdot\delta\bar{\mathbf{n}}_{,\beta} + \Delta\mathbf{u}_{,\beta}\cdot\delta\bar{\mathbf{n}}_{,\alpha}\\ &\quad + \bar{\mathbf{a}}_\alpha\cdot[\Delta(\delta\bar{\mathbf{n}})]_{,\beta} + \bar{\mathbf{a}}_\beta\cdot[\Delta(\delta\bar{\mathbf{n}})]_{,\alpha}\}.\end{aligned} \tag{83}$$

The corresponding incremental work-conjugate stress measures are

$$\begin{aligned}\Delta N^{\alpha\beta} &= \frac{\mathrm{d}}{\mathrm{d}\eta}N^{\alpha\beta}(\eta)|_{\eta=0} = C_1^{\alpha\beta\lambda\mu}\Delta\gamma_{\lambda\mu} + C_2^{\alpha\beta\lambda\mu}\Delta\kappa_{\lambda\mu},\\ \Delta M^{\alpha\beta} &= \frac{\mathrm{d}}{\mathrm{d}\eta}M^{\alpha\beta}(\eta)|_{\eta=0} = C_3^{\alpha\beta\lambda\mu}\Delta\gamma_{\lambda\mu} + C_4^{\alpha\beta\lambda\mu}\Delta\kappa_{\lambda\mu},\end{aligned} \tag{84}$$

where the tangent elasticities $C_n^{\alpha\beta\lambda\mu}$, $n = 1, 2, 3, 4$, are defined through further differentiation of the constitutive equations (55), i.e. the strain energy function Φ given by (40) or (47), with respect to $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$.

$$C_1^{\alpha\beta\lambda\mu} = \frac{\partial^2\Phi}{\partial\gamma_{\alpha\beta}\,\partial\gamma_{\lambda\mu}}, \qquad C_2^{\alpha\beta\lambda\mu} = \frac{\partial^2\Phi}{\partial\gamma_{\alpha\beta}\,\partial\kappa_{\lambda\mu}},$$
$$C_3^{\alpha\beta\lambda\mu} = \frac{\partial^2\Phi}{\partial\kappa_{\alpha\beta}\,\partial\gamma_{\lambda\mu}}, \qquad C_4^{\alpha\beta\lambda\mu} = \frac{\partial^2\Phi}{\partial\kappa_{\alpha\beta}\,\partial\kappa_{\lambda\mu}}. \tag{85}$$

Now, with the help of (62) and (81)–(84) the interior part of (79) can be presented in the form

$$\Delta G_i[\mathbf{u}\,;\Delta\mathbf{u},\delta\mathbf{u}] = \iint_{\mathfrak{M}} (\Delta\sigma_M + \Delta\sigma_G)\,\mathrm{d}A, \tag{86}$$

$$\Delta\sigma_M = \Delta N^{\alpha\beta}\delta\gamma_{\alpha\beta} + \Delta M^{\alpha\beta}\delta\kappa_{\alpha\beta} = \delta\gamma_{\alpha\beta}C_1^{\alpha\beta\lambda\mu}\Delta\gamma_{\lambda\mu} + \delta\gamma_{\alpha\beta}C_2^{\alpha\beta\lambda\mu}\Delta\kappa_{\lambda\mu}$$
$$+\,\delta\kappa_{\alpha\beta}C_3^{\alpha\beta\lambda\mu}\Delta\gamma_{\lambda\mu} + \delta\kappa_{\alpha\beta}C_4^{\alpha\beta\lambda\mu}\Delta\kappa_{\lambda\mu}. \tag{87}$$

$$\Delta\sigma_G = N^{\alpha\beta}\Delta\mathbf{u}_{,\alpha}\cdot\delta\mathbf{u}_{,\beta} + M^{\alpha\beta}\{\Delta\bar{\mathbf{n}}_{,\alpha}\cdot\delta\mathbf{u}_{,\beta} + \Delta\mathbf{u}_{,\alpha}\cdot\delta\bar{\mathbf{n}}_{,\beta} + \bar{\mathbf{a}}_\alpha\cdot[\Delta(\delta\bar{\mathbf{n}})]_{,\beta}\}. \tag{88}$$

In the finite element implementation of the shell theory the term (87), appearing as a result of the linearization of stress measures, allows the construction of the material part of the tangent stiffness matrix at $\mathbf{u}$. The term (88) resulting from the change of shell geometry leads to the geometric part of the tangent stiffness matrix at $\mathbf{u}$.

Analogously we have to derive the Gâteaux differential of the external virtual work (63) leading to:

$$\Delta G_e(\mathbf{u}\,;\Delta\mathbf{u},\delta\mathbf{u}) = \iint_{\mathfrak{M}} \Delta\mathbf{p}\cdot\delta\mathbf{u}\,\mathrm{d}A + \int_{\mathfrak{C}_f} [\Delta\mathbf{T}^*\cdot\delta\mathbf{u} + \Delta\mathbf{H}^*\cdot\delta\bar{\mathbf{n}} + \mathbf{H}^*\cdot\Delta(\delta\bar{\mathbf{n}})]\,\mathrm{d}s, \tag{89}$$

where the incremental values of the external loads are defined by

$$\Delta\mathbf{p} = \frac{\mathrm{d}}{\mathrm{d}\eta}\mathbf{p}(\eta)|_{\eta=0}, \quad \Delta\mathbf{T}^* = \frac{\mathrm{d}}{\mathrm{d}\eta}\mathbf{T}^*(\eta)|_{\eta=0}, \quad \Delta\mathbf{H}^* = \frac{\mathrm{d}}{\mathrm{d}\eta}\mathbf{H}^*(\eta)|_{\eta=0} \tag{90}$$

and the incremental values of $\bar{\mathbf{n}}$ and $\delta\bar{\mathbf{n}}$ at the shell boundary $\mathfrak{C}$ by:

$$\Delta\bar{\mathbf{n}} = \frac{\mathrm{d}}{\mathrm{d}\eta}\bar{\mathbf{n}}[(\eta),n_v(\eta)]|_{\eta=0} = \frac{1}{a_v}[(\mathbf{v}\times\bar{\mathbf{n}})\bar{\mathbf{n}}\cdot\Delta\mathbf{u}' + (\bar{\mathbf{r}}'\times\bar{\mathbf{n}})\Delta n_v]$$

$$\Delta(\delta\bar{\mathbf{n}}) = \frac{\mathrm{d}}{\mathrm{d}\eta}\delta\bar{\mathbf{n}}[\mathbf{u}(\eta),n_v(\eta)\,;\delta\mathbf{u},\delta n_v]|_{\eta=0}$$
$$= -\frac{1}{a_v^2}[(\Delta\mathbf{u}'\times\bar{\mathbf{n}} + \bar{\mathbf{r}}'\times\Delta\bar{\mathbf{n}})\cdot\mathbf{v}][(\mathbf{v}\times\bar{\mathbf{n}})\bar{\mathbf{n}}\cdot\delta\mathbf{u}' + (\bar{\mathbf{r}}'\times\bar{\mathbf{n}})\delta n_v]$$
$$+\frac{1}{a_v}[(\mathbf{v}\times\Delta\bar{\mathbf{n}})\bar{\mathbf{n}}\cdot\delta\mathbf{u}' + (\mathbf{v}\times\bar{\mathbf{n}})\Delta\bar{\mathbf{n}}\cdot\delta\mathbf{u}' + (\Delta\mathbf{u}'\times\bar{\mathbf{n}} + \bar{\mathbf{r}}'\times\Delta\bar{\mathbf{n}})\delta n_v + (\bar{\mathbf{r}}'\times\bar{\mathbf{n}})\Delta(\delta n_v)]. \tag{91}$$

Introducing (91) into the boundary integral of (89) and using some further transformations and integration by parts we can derive an integral expression linear in $\delta\mathbf{n}$, δn_v and $\Delta(\delta n_v)$, where $\Delta(\delta n_v)$ is the second Gâteaux differential of n_v depending non-linearly on $\mathbf{u}$ [see Stumpf (1984, 1986)].

Finally we have to consider the second term of (77) representing the virtual work of the approximation $\mathbf{u}_n^{(i)}$. It vanishes, if $\mathbf{u}_n^{(i)}$ is an equilibrium configuration and for non-equilibrium configurations it is a measure for the unbalanced forces.

9. COMPUTER IMPLEMENTATION AND NUMERICAL APPLICATION

The strain energy density (40) or (47), with corresponding formulae for $\mathbf{W}_{(n)}$ and for the strain measures (25), (46), (56)–(58), is a function of displacements and their first as well as second surface derivatives. Therefore, in the finite element approximation C^1 interelement continuity is required. In order to fulfill this and all other mechanical requirements a triangular high-precision doubly-curved shell finite element with 54 of freedom is selected. The basic characteristics of the element are described by Cowper (1972) and Harte (1982) and developed to the form used in this paper by Nolte and Schieck (1985, 1986) and Schieck (1989). In this element biquintic polynomials are applied as shape functions for all three displacement components. The numerical calculations of the highly non-linear shell problems presented below have been performed on the 2-pipeline CDC Cyber 205 vector processor available at the Ruhr-University of Bochum. In order to analyze the results and to show the performance of the corresponding computer program various examples of shells made of Neo-Hookean and Mooney material have been calculated. The results obtained for the first approximation theory (40) and for the simplest approximation (47) are compared with those given by Libai and Simmonds (1983, 1988) and Taber (1986). Here we present them for three large strain shell problems, whereas further examples with consistently restricted strains and rotations can be found in Schieck (1989).

The following notations are used:

MLU — Mooney material, Large strains, Unrestricted rotations (see eqn (40)),

MLUS — Mooney material, Large strains, Unrestricted rotations, Simplest approximation (see eqn (47)),

MMLS — Mooney material, Moderate strains, Large rotations, Simplest approximation (see Schieck (1989)),

LSU — Linear material, Small strains, Unrestricted rotations (classical geometrically non-linear shell theory for small strains and unrestricted rotations).

The theory variant MMLS is derived from the variant MLUS by assuming moderate membrane strains allowing to simplify (27) to $\lambda \approx 1-\gamma_\alpha^\alpha$ and $\lambda^2 \approx 1-2\gamma_\alpha^\alpha$.

9.1. *Cylindrical shell made of Neo-Hookean material under the action of two line forces applied to diametrically opposite points*

This one-dimensional example was calculated analytically by Libai and Simmonds (1983, 1988) within the large-strain theory of plane rods using several types of approximate one-dimensional theories. When the displacement w of the points where forces are applied is $w/R = 0.5$ the corresponding membrane strains are $\gamma \simeq 0.05$ and the bending strains are $h\kappa \simeq 0.25$. It is seen from Fig. 2a that our results are in good agreement with those of Libai and Simmonds (1988), although for $w/R = 0.5$ the relative deformed thickness under the force becomes $h/\bar{R} \simeq 0.5$.

It follows furthermore from Fig. 2a that the simplest approximation leads to practically identical results as the consistent first approximation theory. In contrast to this the small strain solution LSU gives quite different results.

9.2. *Clamped circular plate made of Neo-Hookean material under pressure load*

This problem was solved analytically by Taber (1986) for different types of boundary conditions. For clamped boundary the behaviour of the plate is such that near the boundary bending strains are important, but when we move towards the center the membrane-type behaviour becomes dominant. It follows from Fig. 3 that our results for both theory variants MLU and MLUS are identical or in good agreement with Taber (1986). After reaching membrane strains of about 3% the moderate strain solution MMLS slightly differs from the large strain solutions MLU and MLUS. Please note that the classical geometrically

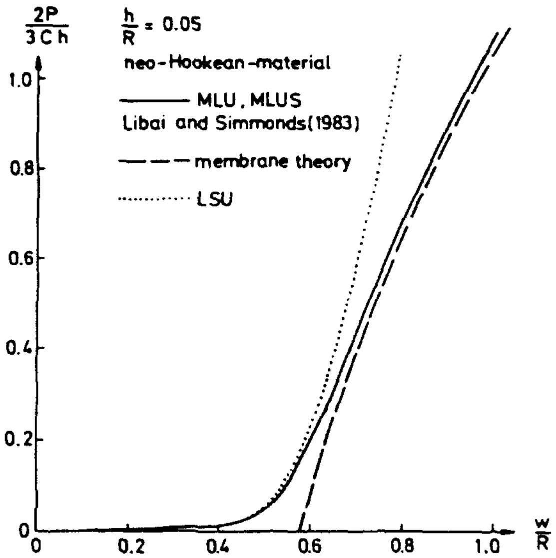

Fig. 2a. Load deflection curves for cylindrical shell subjected to two line forces.

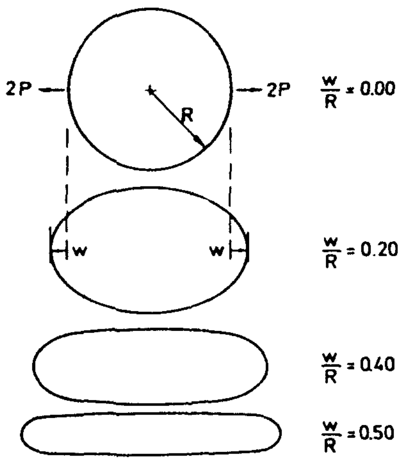

Fig. 2b. Cylindrical shell subjected to two line forces.

non-linear shell theory leads here to observably different results even before the membrane strains reach $\gamma \simeq 0.01$.

9.3. *Clamped circular cylinder made of Mooney material under external pressure*

This two-dimensional highly non-linear shell problem was not yet analyzed in the literature within the large strain range. In order to find easier the buckling mode and to determine the postbuckling behaviour, two small additional forces have been applied to the diametrically opposite points in the middle of the shell. Again both versions MLU and MLUS of our large-strain theory give practically coinciding results, while the classical LSU theory leads to considerably different solution curve (see Fig. 4). The moderate strain solution MMLS starts to diverge considerably from the large strain solution after the membrane strains increased above 3%.

Further examples obtained for doubly curved shells of revolution in Schieck (1989) confirm the results presented here. In particular, in all cases analyzed the difference between

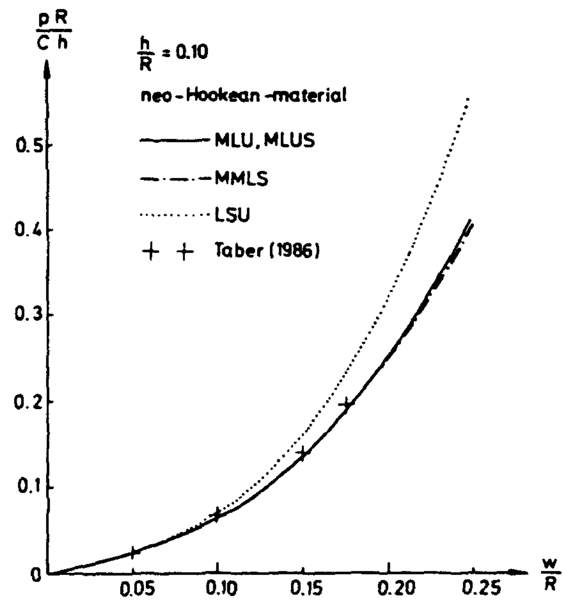

Fig. 3. Load deflection curves for a clamped circular plate under pressure load.

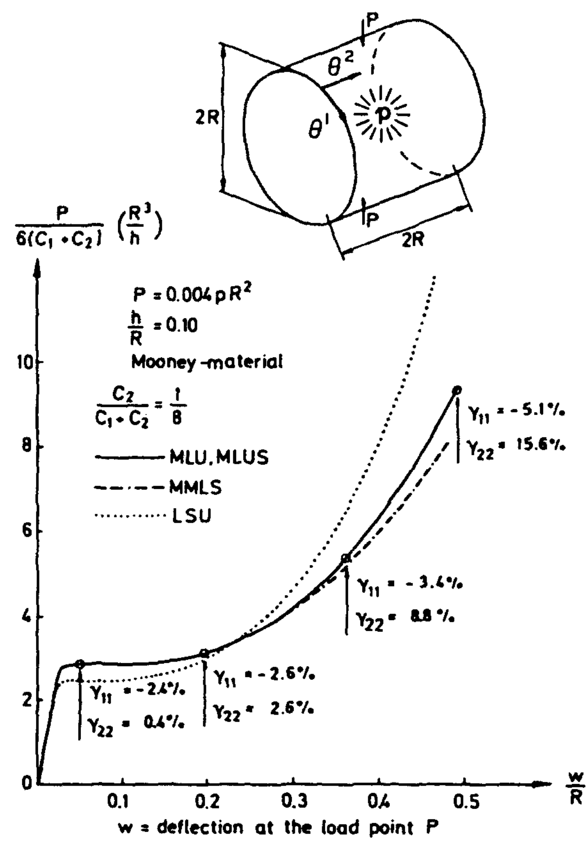

Fig. 4. Load deflection curves for clamped circular cylindrical shell under external pressure.

the MLU and MLUS versions of the theory have remained below 2%. The error of this order has been already introduced into our shell theory by ignoring the transverse shear strains. Therefore, we believe that for engineering analysis of rubber-like shells the use of the simplest approximation (47) to the elastic strain energy function is fully justified.

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